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Some aspects of braided geometry: differential calculus, tangent space, gauge theory

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Abstract. A new approach is suggested to quantum differential calculus on certain quantum varieties. It consists in replacing quantum de Rham complexes with differentials satisfying Leibniz rule by those which are in a sense close to Koszul complexes from Gurevich (1991 *Leningrad Math. J.* **2** 801–28). We also introduce the tangent space on a quantum hyperboloid equipped with an action on the quantum function space and define the notions of quantum (pseudo)metric and quantum connection (partially defined) on it. All objects are considered from the viewpoint of flatness of quantum deformations. The problem of constructing a flatly deformed quantum gauge theory is discussed as well.

1. Introduction

In this paper we consider some problems which can be gathered together under a general name 'braided (quantum, twisted or q-deformed) geometry'. This type of geometry has had a real gold rush since the creation of the quantum group (QG) theory. This phenomenon is motivated by a common desire to generalize methods of ordinary geometry for the needs of mathematical physics since, in accordance with widespread opinion, the future of this discipline is connected with models which are covariant w.r.t. special Hopf algebras rather than to ordinary transformation groups.

Nevertheless, it has turned out that not all objects of the ordinary geometry have their consistent q-analogues. For example, all attempts initiated by Woronowicz [W1, W2] to develop a bicovariant differential calculus on quantum function space $\mathrm{Fun}\,(SL_q(n))$ with two properties (flatness of deformation of the differential algebra† and Leibniz rule for the corresponding differential) have failed.

Moreover, such a differential calculus does not exist. This was shown in [AAM] by considering the corresponding quasiclassical object, namely the graded Poisson–Lie structure, which is an extension of the Sklyanin–Drinfeld bracket to the differential algebra (cf also the last section of [Ar]).

† Let us recall that a deformation $\mathcal{A} \to \mathcal{A}_{\hbar}$ where $\hbar$ is a formal parameter is called *flat* if

1. $\mathcal{A}_{\hbar}/\hbar\mathcal{A}_{\hbar} = \mathcal{A}$
2. $\mathcal{A}_{\hbar}$ and $\mathcal{A}[[\hbar]] = \mathcal{A} \otimes k[[\hbar]]$

are isomorphic as $k[[\hbar]]$-modules (the tensor product is complete in the $\hbar$-adic topology). Here we consider only the objects related to the famous Drinfeld–Jimbo QG $U_q(\mathfrak{g})$. Nevertheless, some of them can be generalized to non-quasiclassical Hecke symmetries, i.e. solutions of the quantum Yang–Baxter equation (QYBE) whose 'symmetric' and 'skew symmetric' algebras possess non-classical Poincaré series (cf [G1]).

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The problem is that a consistent q-deformation of the differential calculus which is well defined on the Lie group $GL(n)$ (more precisely, on the corresponding matrix algebra $\mathrm{Mat}(n)$), cf [T], is not compatible with the constraints resulting from the equation $\det_q = 1$ where $\det_q$ is the quantum determinant. However, some q-deformed differential algebra equipped with a differential without Leibniz rule exists in the $SL(n)$ case, cf [Ar, FP1]. The authors of [FP1] recall a claim of L Faddeev that the Leibniz rule is not reasonable in the quantum case.

One of the main purposes of this paper is to suggest a regular way to construct de Rham-type complexes without any Leibniz rule. The essence of such a complex is close to that of the Koszul complex (of the first kind) introduced in [G1]. We recall the construction of such a complex in section 2. In order to introduce a differential we fix a base in a q-deformed differential algebra and define it only in this base. This saves us from checking the fact that the differential respects the relations which define the algebra in question. Moreover, we realize in the classical case a spectral analysis of de Rham complex, i.e. we study the behaviour of the classical differential on irreducible components of the initial complex and define a quantum differential with similar properties but in the q-deformed category. This approach is realizable when the spectral structure of the complex in question is simple enough (it can be also applied to a non-quasiclassical case).

We apply this approach to a quantum hyperboloid. By construction, its cohomology is just the same as in the classical case. (In general, the following conjecture seems to be very plausible: once a quantum de Rham complex is constructed in a proper way it has for a generic q the same cohomology as its classical counterpart, cf for example [FP2, HS] for an illustration of this conjecture.) The corresponding construction is described in section 3.

In this connection we also discuss the following problem: what is a proper definition of the tangent space on the quantum hyperboloid (in other words, what is the phase space corresponding to the quantum hyperboloid considered as a configuration space)?

We introduce such a tangent space $T(H_q)$ (which is treated as an A-module where A is the quantum function space in question) and equip it with an action

$$T(H_q) \otimes A \to A$$

converting elements of the tangent space into 'braided vector fields'. Let us remark that our construction of braided vector fields is realized without (once more!) any Leibniz rule (cf [A]).

We also introduce (in section 4) the notions of a (pseudo)metric and a connection (partially defined) on the tangent space on the quantum hyperboloid. In all our constructions we impose only two properties on any q-deformed object in question: $U_q(sl(2))$-covariance and flatness of the deformation.

In section 5 we consider the problem of constructing a quantum gauge theory from this viewpoint. In spite of numerous attempts to generalize the classical gauge theory from the above viewpoint, up to now this has not been satisfactory. We are rather sceptical about the possibility of introducing a consistent q-deformation to the classical gauge theory. We discuss this in section 5.

Throughout the paper the basic field k is $\mathbb{R}$ or $\mathbb{C}$ and the parameter $q \in k$ is assumed to be generic.

2. De Rham and Koszul complexes: comparative description

First, let us consider some complexes related to the QG $U_q(sl(n))$. The most popular complexes of such a type are de Rham complexes connected with the first fundamental modules of the QG $U_q(sl(n))$ [WZ] and those defined on the q-deformed matrix algebra $\mathrm{Mat}(n)$ [T]. Whereas the former is one-sided $U_q(sl(n))$-covariant, the latter is bicovariant. (Such complexes exist for any Hecke symmetry: see footnote, section 1.)

A de Rham complex related to the first fundamental $U_q(SO(n))$-module was constructed in [CSW].

However, all the above complexes are, in a sense, objects of quantum (braided or $q-$) linear algebra rather than of quantum geometry. Quantum geometry deals with quantum varieties different from vector spaces: a typical example is $SL_q(n)$ defined by the equation $\det_q = 1$ (by an abuse of the language we speak about a variety although in fact we deal with the corresponding 'quantum function space'). As we said above, the quantum differential calculus well defined on the vector space $\mathrm{Mat}(n)$ cannot be restricted to the variety in question if we want it to be a flat deformation of its classical counterpart and its differential to obey the Leibniz rule. We refer the reader to [I] where this problem is discussed.

Another interesting class of varieties connected with the QG $U_q(\boldsymbol{g})$ are quantum homogeneous spaces which are one-sided $U_q(\boldsymbol{g})$-modules. The products in the corresponding algebras are assumed to be $U_q(\boldsymbol{g})$-covariant:

$$Z(a \cdot b) = Z_{(1)}a \cdot Z_{(2)}b \qquad Z \in U_q(\boldsymbol{g}) \quad Z_{(1)} \otimes Z_{(2)} = \Delta(Z).$$

A quantum homogeneous space is usually introduced via a couple of QG in the spirit of a homogeneous space G/H. However, it is desirable for it to have more explicit description by a system of equations.

An attempt to find such a system for certain q-deformed $SL(n)$-orbits in $sl(n)^*$ featured in [DGK]: in fact, a two-parameter family of quantum algebras was constructed and the problem of which algebra of this family could be considered as a q-analogue of a commutative algebra was not so evident. In the following we consider a particular case of these q-deformed orbits, namely that related to $U_q(sl(2))$ and called *quantum hyperboloid*. Being equipped with a proper involution it becomes Podles' *quantum sphere* [P1] (more precisely a particular case of Podles' quantum sphere which is simply the 'q-commutative' case; note that in this low-dimensional case there is no problem with understanding 'q-commutativity').

The first attempt to construct a q-deformed differential calculus on a quantum sphere was undertaken in [P2]. However, the corresponding differential algebra is not a flat deformation of its classical counterpart. In section 3 we present another approach to introducing a quantum de Rham complex with the flatness property.

Let us evoke now another type of complex connected (in particular) with the QYBE, namely Koszul complexes (we are still working in the framework of quantum linear algebra). Let V be a vector space over k and $I \subset V^{\otimes 2}$ be a subspace of $V^{\otimes 2}$. Let us set

$$I^{(0)} = k \qquad I^{(1)} = V$$
$$I^{(n)} = I \otimes V^{\otimes(n-2)} \cap V \otimes I \otimes V^{\otimes(n-3)} \cap \ldots \cap V^{\otimes(n-2)} \otimes I \qquad n \geqslant 2$$

and consider the quadratic algebra

$$A = T(V)/\{I\} \qquad \text{where} \quad \{I\} \text{ is the ideal generated by } I$$

and $T(V)$ stands for the free tensor algebra of the space V.

Let $A^{(n)}$ be its homogeneous component of degree n. Note that $A^{(0)} = k$, $A^{(1)} = V$ and $A^{(n)}$, $n \geqslant 2$ can be treated as the quotient

$$V^{\otimes n}/I^n \qquad \text{where} \quad I^n = I \otimes V^{\otimes(n-2)} + V \otimes I \otimes V^{\otimes(n-3)} + \cdots + V^{\otimes(n-2)} \otimes I.$$

Then the corresponding Koszul complex is defined by

$$d : A \otimes I^{(n)} \to A \otimes I^{(n-1)} \qquad d(a \otimes x \otimes y)$$
$$= ax \otimes y \qquad \text{where} \quad a \in A \quad x \otimes y \in V \otimes V^{\otimes(n-1)} \tag{2.1}$$

and ax is the product in the algebra A. In fact, this complex decomposes into a series of subcomplexes

$$A^{(m)} \otimes I^{(n)} \to A^{(m+1)} \otimes I^{(n-1)}.$$

Definition 1. *A quadratic algebra A is called Koszul if the cohomology of the complex (2.1) vanishes in all terms (except of course the trivial term $A^{(0)} \otimes I^{(0)}$ consisting of constants, i.e. elements of k).*

Let us suppose now that we have two nontrivial complementary subspaces $I_+ \subset V^{\otimes 2}$ and $I_- \subset V^{\otimes 2}$, i.e. such that $I_+ \cap I_- = \emptyset$ and $I_+ \oplus I_- = V^{\otimes 2}$ and associate to them two algebras

$$A_+ = T(V)/\{I_-\} \qquad \text{and} \qquad A_- = T(V)/\{I_+\}$$

(they are treated as 'symmetric' and 'skew symmetric' algebras whereas the elements of the subspaces $I_\pm \subset V^{\otimes 2}$ are treated as 'symmetric' and 'skew symmetric' tensors). Then we can define two Koszul complexes

$$d : I_+^{(n)} \otimes A_- \to I_+^{(n-1)} \otimes A_- \qquad \text{and} \qquad \delta : A_+ \otimes I_-^{(n)} \to A_+ \otimes I_-^{(n-1)}$$

as described above.

If, moreover, we can identify $A_+^{(n)}$ with $I_+^{(n)}$ and $A_-^{(n)}$ with $I_-^{(n)}$ (this means that the spaces $I_+^{(n)}$ and I_-^n on the one hand and $I_-^{(n)}$ and I_+^n on the other hand are complementary for $n \geqslant 3$, cf [DS]) we can consider these two complexes as one (whose terms are $A_+^{(m)} \otimes A_-^{(n)}$) but equipped with two differentials mapping in opposite directions.

This is just the case of complexes constructed in [G1] (where V is a vector space equipped with a Hecke symmetry, see footnote section 1). As shown in [G1], the algebras $A_\pm$ are Koszul. In particular, this implies the classical relation

$$P_+(t)\, P_-(-t) = 1$$

between the Poincaré series of the 'symmetric' and 'skew symmetric' algebras.

Let us remark that the above identification $A_\pm^{(n)} \approx I_\pm^{(n)}$ was realized in [G1] by means of projectors

$$P_\pm^n : V^{\otimes n} \to I_\pm^{(n)}$$

whose kernels are just $I_\mp^n$. This implies that the spaces $I_\pm^{(n)}$ and $I_\mp^n$ are complementary. Moreover, the differentials d and δ are realized in [G1] directly in terms of these projectors.

We say that an element $x \otimes y \in A_+^{(m)} \otimes A_-^{(n)}$ is given in a canonical (or base) form if $x \otimes y \in I_+^{(m)} \otimes I_-^{(n)}$, i.e. it is realized as a sum of products of 'symmetrized' and 'skew symmetrized' elements. In virtue of [G1] any element of $A_+^{(m)} \otimes A_-^{(n)}$ can be represented in a canonical form.

Let us now compare these complexes with the de Rham complex constructed in [WZ]. By applying the de Rham differential to the product $x_{i_1}x_{i_2}\ldots x_{i_m}$ one obtains, by virtue of the Leibniz rule, a sum whose arbitrary summand is of the form

$$x_{i_1}x_{i_2}\ldots x_{i_{p-1}}dx_{i_p}\,x_{i_{p+1}}\ldots x_{i_m} \qquad 1 \leqslant p \leqslant m.$$

(The sign $\otimes$ is systematically omitted.) Here $\{x_i\}$ is a base of the space V equipped with a Yang–Baxter operator of the Hecke type.

The second step of the procedure consists of moving the factor dx_{i_p} to the right side (for concreteness). So, the problem arises of finding a moving which would be compatible with the differential and would lead to a flat deformation of the initial differential algebra. If, moreover, one wants to restrict the differential to a quantum variety it is necessary to coordinate such a movement with constrains arising from the system of equations defining the variety in question.

Nevertheless, such a problem does not appear for the Koszul complex (2.1) since its differential d takes only one (namely, extreme) factor of the space $I^{(n)}$ to the algebra A. So, one should not transpose the elements from V and their differentials.

We can say that the Koszul complex from [G1] and the de Rham one from [WZ] are formed by the same terms. The difference is that all elements of the Koszul complex are represented in the canonical form. Moreover, it is easy to see that the differentials of these two complexes are proportional to each other on each term (and the coefficients are not trivial). This implies that their cohomologies are isomorphic (recall that q is generic).

Since the cohomology of the Koszul complex is trivial (apart from the $(0, 0)$ term) we obtain that it is also true for de Rham complex from [WZ] (a quantum version of the Poincaré lemma).

Let us remark that this scheme can be extended to other couples of subspaces $I_\pm$ associated to the QYBE (including non-quasiclassical cases) but the crucial problem is to show that the associated spaces $I_+^{(n)}$ and I_-^n (resp., $I_-^{(n)}$ and I_+^n) are complementary (cf [DS]).

3. De Rham-type complex on quantum hyperboloid

Let us pass now to a quantum hyperboloid. Consider the QG $U_q(sl(2))$ generated by the generators X, H, Y subject to the well known relations (cf [CP]). Let us fix a coproduct and the corresponding antipode and consider the spin 1 $U_q(sl(2))$-module $V = V^q$.

In order to define a quantum hyperboloid we should fix a base in V and write down the system of relations on the generators compatible with action of the QG in question. However, we want to represent this system in a symbolic way without referring to its specific coordinate form.

We need only the fact that the fusion ring for $U_q(sl(2))$-modules is exactly the same as in the classical case (we consider only the finite-dimensional $U_q(sl(2))$-modules which are deformations of the $sl(2)$-modules). Thus, if V_i is the spin i $U_q(sl(2))$-module then the classical formula

$$V_i \otimes V_j = \oplus_{k=|i-j|}^{i+j} V_k$$

is still valid although the Clebsch–Gordan coefficients (which depend on a base) are q-deformed.

In particular, we have

$$V^{\otimes 2} = V_0 \oplus V_1 \oplus V_2.$$

We keep the notation V for the initial space and V_1 for the component in $V^{\otimes 2}$ isomorphic to V. Let us fix in the spaces V, V_0, V_1 and V_2 some highest weight (h.w.) elements v, v_0, v_1 and v_2, respectively, and impose the relations (which are the most general relations compatible with action of the QG $U_q(sl(2))$)

$$v_0 = c \qquad v_1 = \overline{h} v. \tag{3.1}$$

Here $c \in k$ and $\overline{h} \in k$ are some constants. One can now deduce the complete system of equations by applying to the second relation the decreasing operator $Y \in U_q(sl(2))$.

Let us denote $\mathcal{A}^c_{\hbar\, q}$ as the algebra defined by (3.1) and derivative relations.

This algebra possesses the following property: it is multiplicity free. More precisely, any integer spin module occurs once in its decomposition into a direct sum of irreducible $U_q(sl(2))$-modules. Moreover, any element of $\mathcal{A}^c_{\hbar\, q}$ can be represented in a unique way as a sum of homogeneous elements belonging to the components $V_i \subset V^{\otimes i}$ (a proof of this fact can be deduced, for example, from [GV]). This representation will be called *canonical or base*. Note that element $v^{\otimes i}$ is a h.w. one of the component V_i.

We treat a particular case of the algebra in question, namely $\mathcal{A}^c_{0\,q}$, as a q-analogue of a commutative algebra and call it *quantum hyperboloid* if $c \neq 0$ and *quantum cone* if $c = 0$. Since the $sl(2)$-module $sl(2)^{\otimes 2}$ is multiplicity free we can introduce q-analogues $I_\pm$ of symmetric an skew symmetric subspaces of $sl(2)^{\otimes 2}$ by setting similarly to the classical case

$$I_+ = V_0 \oplus V_2 \qquad \text{and} \qquad I_- = V_1.$$

Let us emphasize that the corresponding algebras $A_\pm = T(sl(2))/\{I_\mp\}$ are flat deformations of their classical counterparts.

We will need also a q-deformed (braided) Lie bracket. It can be defined as a non-trivial map

$$[\ ,\ ]_q : V^{\otimes 2} \to V$$

($V = sl(2)$ as linear spaces) being a $U_q(sl(2))$-morphism. By this request the bracket is defined in a unique way up to a factor.

Remark 1. *Let us remark that for the Lie algebras $\boldsymbol{g} = sl(n)$, $n \geqslant 3$ the $\boldsymbol{g}$-module $\boldsymbol{g}^{\otimes 2}$ is not multiplicity free any more: it possesses two components isomorphic to $\boldsymbol{g}$ itself: one belongs to the symmetric part of $\boldsymbol{g}^{\otimes 2}$ and the other one to the skew symmetric part. This is the reason why it is not so evident what are q-analogues of the symmetric and skew symmetric algebras of the space $\boldsymbol{g}$. However, there exists a subspace $I_- \subset \boldsymbol{g}_q^{\otimes 2}$ where $\boldsymbol{g}_q = sl(n)$ as vector spaces but equipped with a $U_q(sl(n))$-module structure such that the quadratic algebra $T(\boldsymbol{g}_q)/\{I_-\}$ is a flat deformation of the symmetric algebra of $\boldsymbol{g}$ (cf [D]). A more explicit description of I_- can be given by means of the so-called reflection equation (RE)*

$$SL_1SL_1 - L_1SL_1S = 0 \tag{3.2}$$

where S is a solution of the QYBE (here of the Hecke type), $L_1 = L \otimes \mathrm{id}$ and L is a matrix with matrix elements (l_i^j). The quadratic algebra defined by the system (3.2) is usually called RE algebra.

Let us remark that the RE algebra is covariant w.r.t. $U_q(sl(n))$ (cf [IP]) and it is a flat deformation of its classical counterpart $\mathrm{Sym}(W)$ where $W = \mathrm{span}\,(l_i^j)$ (cf [L]). It is not difficult to see that the space W is a sum of two irreducible $U_q(\boldsymbol{g})$-modules: one-dimensional one with a generator $l = \mathrm{tr}_q\, L$ where tr_q is the q-trace and $n^2 - 1$-dimensional one which can be identified with $\boldsymbol{g}_q$ above. By killing the component l (i.e. by passing to the quotient of the RE algebra over the ideal $\{l\}$) we get exactly algebra mentioned above, $T(\boldsymbol{g}_q)/\{I_-\}$. In other words, the space I_- is defined by the relation (3.2) but with one component less.

Moreover, by means of the RE algebra one can get an algebra looking like the enveloping algebra of q-deformed Lie algebra $sl(n)$ introduced in [LS]. Before killing the component l let us realize a shift $l_i^j \to l_i^j + \overline{h}\delta_i^j$. Then instead of a graded quadratic algebra we get a filtered algebra, defined by quadratic-linear relations. Now by killing l we get a quadratic-linear algebra with $n^2 - 1$ generators. This is just another realization of the enveloping algebra from [LS] (if in the latter algebra we replace the Casimir element by a constant, cf [LS]) and a flat two-parameter deformation of $\mathrm{Sym}(\boldsymbol{g})$ whose existence was stated in [D]. (However, to get a reasonable quasiclassical limit we should replace the parameter $\overline{h}$ in this quadratic-linear algebra by $\overline{h}/(q-1)$.)

If $\boldsymbol{g}$ is a simple Lie algebra different from $sl(n)$ its tensor square is multiplicity free. This allows one to define a q-deformed Lie bracket requiring it to be a non-trivial morphism in the category of $U_q(\boldsymbol{g})$-modules (this defines the bracket uniquely up to a factor) and to introduce the enveloping algebra of the corresponding 'braided Lie algebra' $\boldsymbol{g}_q$. Deformed analogues of the symmetric and skew symmetric algebras of the space $\boldsymbol{g}$ are also well defined. However, these algebras are not flat deformations of their classical counterparts (cf [G2]).

Remark 2. *Let us emphasize that the classical counterpart $\mathcal{A}^c_{0\,1}$ of the algebra $\mathcal{A}^c_{0\,q}$ contains only polynomials restricted to the hyperboloid (or the cone). This is the reason why its properties and those of the function algebra on the sphere are similar: a passage from one algebra to the other one can be realized by a change of base. For example, this passage does not change the cohomology of the de Rham complex (see below).*

Let us set $\Omega^0 = \mathcal{A}^c_{0\,q}$. Our next step is to define the spaces of first- and second-order differential forms over this algebra. First, consider the tensor products

$$\wedge^1 = \mathcal{A}^c_{0\,q} \otimes V' \qquad \text{and} \qquad \wedge^2 = \mathcal{A}^c_{0\,q} \otimes V_1''.$$

Their second factors are treated as pure differentials (say the element $x \otimes y \in \wedge^1$ is treated as $x\,\mathrm{d}y$ and in $x \otimes y \in \wedge^2$ the factor $y \in V_1''$ is a sum of products of two pure first-order differentials). The mark $'$ stands for a pure first-order differential term and that $''$ stands for a pure second-order differential term. Thus, the space V' (resp. V_1'') is isomorphic to the space V_1 itself; the isomorphism is defined by

$$\mathrm{d}x_i \to x_i (\text{resp.}\, \mathrm{d}x_i \otimes \mathrm{d}x_j \to x_i \otimes x_j).$$

Note that we treat the vector spaces $\wedge^i$ as left $\mathcal{A}^c_{0\,q}$-modules. We do not endow their sum $\oplus \wedge^i$ with any algebraic structure. So, we do not need any transposition rule for the elements of $\mathcal{A}^c_{0\,q}$ and their differentials.

Let us introduce now the first- and second-order differential forms on the quantum hyperboloid by

$$\Omega^1 = \wedge^1/\{(V \otimes V')_0\} \qquad \Omega^2 = \wedge^2/\{(V \otimes V_1'')_1 + (v_0 - c) \otimes V_1''\}.$$

Here the terms in the denominators are not ideals but only left $\mathcal{A}^c_{0\,q}$-submodules of $\wedge^1$ and $\wedge^2$, respectively. The notation $(V \otimes V')_i$ means that in the product $V \otimes V'$ we take the spin i component (similarly for $(V \otimes V_1'')_i$). And $(v_0 - c) \otimes V_1''$ stands for the second-order differential forms containing $v_0 - c$ as a factor.

To make this construction more explicit let us represent it in a base form (by restricting ourselves to the classical case since it does not matter what case, classical or quantum we deal with). Let u, v, w be the usual base in $\mathrm{Fun}\,(sl(2)^*) = \mathrm{Sym}(sl(2))$. Then the above denominators are generated respectively by

$$2u\,\mathrm{d}w + 2w\,\mathrm{d}u + v\,\mathrm{d}v$$

and

$$2ue_2 + ve_1 \qquad ue_3 - we_1 \qquad 2we_2 + ve_3 \qquad (2uw + 2wu + vv - c)e_i \qquad i = 1, 2, 3$$

$$\text{with} \quad e_1 = \mathrm{d}u\,\mathrm{d}v - \mathrm{d}v\,\mathrm{d}u \quad e_2 = \mathrm{d}u\,\mathrm{d}w - \mathrm{d}w\,\mathrm{d}u \quad e_3 = \mathrm{d}v\,\mathrm{d}w - \mathrm{d}w\,\mathrm{d}v$$

(we suppose here that $v_0 = 2uw + 2wu + vv$).

We have defined the spaces Ω^i, $i = 1, 2$ as some quotients. Now, we want to define the differentials in some bases of these spaces similarly to the Koszul complexes discussed above. To define such bases we realize a spectral analysis of the spaces Ω^i, i.e. decompose these spaces into a direct sum of irreducible $sl(2)$-modules. First, describe the components in the products

$$V \otimes V' \qquad V_i \otimes V' \qquad V_i \subset \mathcal{A}^c_{0\,1} \qquad i = 2,\ 3, \ldots$$

which are surviving in the quotient space Ω^1. It is evident that in the product $V \otimes V'$ only the components $(V \otimes V')_1$ and $(V \otimes V')_2$ survive since by construction the component $(V \otimes V')_0$ is equal to zero in the quotient.

By a similar reason in the product $V_2 \otimes V'$ the components $(V_2 \otimes V')_2$ and $(V_2 \otimes V')_3$ survive and that $(V_2 \otimes V')_1$ is equal to zero modulo the terms of $k \otimes V' = V'$. This can be explained as follows. The elements of $\mu^{1,2}(V \otimes (V \otimes V')_0)$ are trivial in Ω^1 by construction. Here μ stands for the product in $\mathcal{A}^c_{0\,q}$, the indices, 1, 2 mean, as usual, that the operator μ is applied to the first two factors. By reducing any element of the product $V \otimes V$ to the canonical form we get a sum of an element from $V_2 \subset V^{\otimes 2}$ and another one from k. This completes the proof.

Similarly, in the product $V_i \otimes V'$ the component $(V_i \otimes V')_{i-1}$ is equal to zero modulo the terms belonging to $V_j \otimes V'$, $j < i$. Thus, we have shown the following.

Proposition 1. *The base in the $\mathcal{A}^c_{0\,q}$-module Ω^1 is formed by*

$$(1)V' \qquad (2)(V \otimes V')_{1,2} \qquad (3)(V_2 \otimes V')_{2,3} \qquad (4)(V_3 \otimes V')_{3,4} \quad \text{etc.}$$

In a similar way one can perform a spectral analysis of the $\mathcal{A}^c_{0\,q}$-module Ω^2 and describe its base.

Proposition 2. *The base in the $\mathcal{A}^c_{0\,q}$-module Ω^2 is formed by*

$$(1)V_1'' \qquad (2)(V \otimes V_1'')_{0,2} \qquad (3)(V_2 \otimes V_1'')_3 \qquad (4)(V_3 \otimes V_1'')_4 \quad \text{etc.}$$

An evident difference between the modules Ω^1 and Ω^2 consists in the following. The module Ω^2 is defined as a quotient of $\wedge^2$ over the sum of two submodules. Therefore, two components in the products $V_i \otimes V_1''$, $i \geqslant 2$ disappear and only one survives. The component $V_1 \otimes V'$ is exceptional because the relation $v_0 = c$ does not lead to any constraint for it. Let us consider now the de Rham complex in the classical case

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \longrightarrow 0. \tag{3.3}$$

Since the differential commutes with the $sl(2)$ action it takes any irreducible $sl(2)$-module to either an isomorphic $sl(2)$-module or 0. Using propositions 1 and 2 it is not difficult to describe the irreducible $sl(2)$-modules of Ω^i, $i = 0, 1, 2$ belonging to $\operatorname{Ker} d$ and those belonging to $\operatorname{Im} d$.

Proposition 3. *1. In Ω^0 the only trivial module, i.e. that consisting of the elements of k belongs to* Ker d_0.

$$2.\operatorname{Ker} d_1 = V' \oplus (V \otimes V')_2 \oplus (V_2 \otimes V')_3 \oplus (V_3 \otimes V')_4 \oplus \ldots$$

and therefore the modules

$$(V \otimes V')_1 \qquad (V_2 \otimes V')_2 \qquad (V_3 \otimes V')_3 \ldots$$

go to isomorphic modules in Ω^2.

Corollary 1. *The cohomology of the complex (3.3) is the following:*

$$\dim H^0 = 1 \qquad \dim H^1 = 0 \qquad \dim H^2 = 1.$$

H^0 is generated by 1 and H^2 is generated by $(V \otimes V_1'')_0$.

Thus, it is just the cohomology of the sphere (see above, remark 2).

Let us now extend the de Rham complex (3.3) to the quantum case. The terms of the quantum complex have just the same irreducible components as their classical counterparts (but these components become $U_q(sl(2))$-modules). Now we should define differentials. Define them on each term by requiring them to be $U_q(sl(2))$-morphisms and to be flat deformations of the classical differentials (by this demand the differentials are defined on each $U_q(sl(2))$-module in a unique way up to a factor). Finally, we have by construction exactly the same cohomology as in the classical case.

Comparing our construction with that from [P2] we repeat that the latter is not any flat deformation of its classical counterpart while our deformation is flat by construction. On the other hand, we have lost the structure of an algebra in the $\mathcal{A}^c_{0\,q}$-module $\Omega = \oplus\Omega^i$ and the Leibniz rule for the differentials.

4. Quantum tangent space and related structures

In this section we introduce the tangent space on quantum hyperboloid and discuss some derived structures (metric, connection). We also discuss a way to realize the q-deformed tangent space by means of 'braided vector fields'. Hopefully, this approach is valid for other quantum varieties like quantum orbits considered in [DGK]. As in the previous section we avoid using any specific base form.

First, we consider a sphere S^2 given by $x^2 + y^2 + z^2 = c$. Let Fun (S^2) be the space of the polynomials restricted to the sphere and Vect (S^2) be the space of left vector fields. The latter space is generated as a left Fun (S^2)-module by three infinitesimal rotations

$$X = y\partial_z - z\partial_y \qquad Y = z\partial_x - x\partial_z \qquad Z = x\partial_y - y\partial_x.$$

It is easy to check that the vector fields X, Y, Z satisfy the following relation:

$$x\,X + y\,Y + z\,Z = 0 \tag{4.1}$$

(x, y, z are treated here as operators via the product operator in the algebra Fun (S^2)). So, as a Fun (S^2)-module Vect (S^2) can be realized as the quotient M/N where

$$M = \{a\,X + b\,Y + c\,Z,\ a, b, c \in \mathrm{Fun}\,(S^2)\}$$
$$N = \{f\,(x\,X + y\,Y + z\,Z),\ f \in \mathrm{Fun}\,(S^2)\}.$$

In what follows we call this Fun (S^2)-module *tangent space* and denote it $T(S^2)$. In fact it is simply the Vect (S^2) space but we want to emphasize by this notation that we ignore the operator meaning of this space. As usual, the tangent space is introduced in local terms as a vector bundle. However, in the quantum case such a local description is not possible.

In a similar way there can be introduced the tangent space $T(H)$ on a hyperboloid H. Namely, it can be realized as the quotient of a free $\mathcal{A}^c_{0\,1}$-module M over its submodule

$$N = \{f\,(2u\,W + 2w\,U + v\,V)\}.$$

This is also motivated by the operator meaning of the generators: the generators U, V, W are represented in the algebra Fun $(H) = \mathcal{A}^c_{0\,1}$ by infinitesimal hyperbolic rotations.

Note that the symmetric algebra of the tangent space $T(S^2)$ can be treated as the function algebra on the underlying four-dimensional algebraic variety embedded in the six-dimensional space

$$(\mathrm{span}\,(x, y, z, X, Y, Z))^*$$

this variety is defined by the equation of the sphere and that (4.1) (if $k = \mathbb{R}$ it is true for $c > 0$). A similar description is also valid for the symmetric algebra of $T(H)$.

Unfortunately, there does not exist any quantum analogue of this algebra being its flat deformation (see below). Nevertheless, a reasonable q-deformation of tangent space equipped with an appropriated module structure exists. The aim of this section is to describe this deformation, i.e., to introduce the tangent space on the quantum hyperboloid as an $\mathcal{A}^c_{0\,q}$-module and to realize its elements as operators looking like vector fields on the classical object.

In order to do it we represent the defining relation of the tangent space $T(H)$ in a symbolic way:

$$(V \otimes V')_0 = 0 \tag{4.2}$$

(hereafter the mark $'$ designs the space span (U, V, W)). We treat the tangent space on the hyperboloid as a left $\mathcal{A}^c_{0\,1}$-module (as a right $\mathcal{A}^c_{0\,1}$-module the tangent space can be given by $(V' \otimes V)_0 = 0$).

It is evident that if we want to define the tangent space on the quantum hyperboloid as a flat deformation of its classical counterpart we should use the same formula (4.2) but in the category of $U_q(sl(2))$-modules. Let us be a precise. First, we introduce the left $\mathcal{A}^c_{0\,q}$-module $\wedge^1$ as in the previous section but with another signification of the space V'. This means that the generators $\mathrm{d}u,\ \mathrm{d}v,\ \mathrm{d}w$ are replaced by $U,\ V,\ W$, while the $U_q(sl(2))$-module structure of V' is unchanged. Second, we define the *tangent space* on the quantum hyperboloid as its quotient like Ω^1 above (fortunately, both the tangent and cotangent spaces as $\mathcal{A}^c_{0\,q}$-modules are defined by the same equation (4.2)). Let us denote the quotient object by $T(H_q)$ reserving the notation H_q for the quantum hyperboloid.

Proposition 4. *The $\mathcal{A}^c_{0\,q}$-module $T(H_q)$ is a flat deformation of its classical counterpart.*

Proof follows immediately from the explicit construction of the base of this quotient given in the previous section.

Let us now assign an operator meaning to the elements of the space $T(H_q)$.

Proposition 5. *There exists a map*

$$\beta : T(H_q) \otimes \mathcal{A}^c_{0\,q} \to \mathcal{A}^c_{0\,q} \tag{4.3}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A}^c_{0\,q} \otimes T(H_q) \otimes \mathcal{A}^c_{0\,q} & \longrightarrow & T(H_q) \otimes \mathcal{A}^c_{0\,q} \\ \downarrow & & \downarrow \\ \mathcal{A}^c_{0\,q} \otimes \mathcal{A}^c_{0\,q} & \longrightarrow & \mathcal{A}^c_{0\,q} \end{array}$$

is associative. Here the elements of $\mathcal{A}^c_{0\,q}$ act on $\mathcal{A}^c_{0\,q}$ (the low arrow) by the usual product. The vertical arrows are defined by means of β and the top one makes use of the $\mathcal{A}^c_{0\,q}$-module structure of $T(H_q)$. (Thus, the map β realizes an action of the space $T(H_q)$ on the algebra $\mathcal{A}^c_{0\,q}$.)

This proposition allows us to realize the tangent space as an operator algebra where the elements of the algebra $\mathcal{A}^c_{0\,q}$ act via the product operator. We call the elements of the space $T(H_q)$ (left) *braided vector fields* if the operators $\beta(V')$ satisfy the relations

$$(\beta \otimes \beta)(V' \otimes V')_1 - \sigma\, \beta\, [\ ,\ ]_q (V' \otimes V')_1 = 0$$

where $[\ ,\ ]_q$ is the q-deformed Lie bracket introduced in section 3 and $\sigma \in k$ is a non-trivial factor. This means that β realizes a representation (in the sense of [LS]) of the braided Lie algebra defined by the bracket $\nu[\ ,\ ]_q$ with a proper factor ν.

Proposition 6. *There exists a map β from the previous proposition such that the elements of $T(H_q)$ being represented via β becomes braided vector fields.*

We refer the reader to [A] for proofs of these statements (the main idea of the construction has been suggested in [DG2]). Here, we only want to say that the problem is to find good candidates for the role of q-analogues of the infinitesimal hyperbolic rotations $U,\ V,\ W$. They arise from the adjoint action of the q-Lie algebra $sl(2)_q$ onto itself (note that the operators $X,\ H,\ Y$ coming from the QG $U_q(sl(2))$ do not satisfy the relation (4.2)).

Let us remark that similar statements are valid for the tangent space treated as a right $\mathcal{A}^c_{0\,q}$-module.

Thus, we have an embedding

$$sl(2)_q \hookrightarrow T(H_q) \tag{4.4}$$

where the tangent space is realized as braided vector fields space. This embedding is a deformation of its classical counterpart which is the simplest example of a so-called anchor (recall that an anchor consists of an variety M, a Lie algebra $\boldsymbol{g}$ and an embedding of $\boldsymbol{g}$ into the vector field space on M). This is why we call the embedding (4.4) *quantum anchor* in spite of the fact that the whole of the space $T(H_q)$ is not equipped with any q-deformed Lie bracket. We consider also the data $(T(H_q), \mathcal{A}^c_{0\,q})$ as a partial q-analogue of Lie–Rinehart algebras [R] ('partial' means here that the space $T(H_q)$ is not equipped with any 'q-Lie algebra' structure properly coordinated with the product operator in the algebra $\mathcal{A}^c_{0\,q}$).

After having represented the space $T(H_q)$ by braided vector fields it is natural to introduce the space of braided differential operators as that generated by the braided vector fields and the elements of $\mathcal{A}^c_{0\,q}$ treated as zero-order operators (see above). In the classical case this space is spanned by the subspaces

$$\mathcal{A}^c_{0\,1} \otimes V'^{\otimes n}.$$

The fact that this algebra is closed w.r.t. the operator product is assured by the Leibniz rule: by means of this rule it is possible to represent a product of two elements of this form as a linear combination of such elements.

Unfortunately, in the quantum case any form of the Leibniz rule does not exist (this fact can be checked by direct calculations). Roughly speaking, this means that there does not exist any reasonable way to transpose the elements of the algebra $\mathcal{A}^c_{0\,q}$ and those of the space V'. This is also the reason why there does not exist any 'q-symmetric algebra' of the quantum tangent space $T(H_q)$ being a flat deformation of its classical counterpart (see above). Without going into detail we say only that the Yang–Baxter operator (arising from the universal R-matrix) being at first glance a good candidate for the role of such a transposition leads to a non-flat deformation of the classical symmetric algebra. (See also below, remark 3).

Let us pass now to the problem of constructing a q-deformed metric on the quantum tangent space. To distinguish the quantum tangent spaces equipped with the left and right $\mathcal{A}^c_{0\,q}$-module structures we will use for the first (second) one the notation $T(H_q)_l$ ($T(H_q)_r$).

Definition 2. *We say that an operator*

$$\langle , \rangle : T(H_q)_l \otimes_k T(H_q)_r \to \mathcal{A}^c_{0\,q}$$

is quantum (pseudo-)metric if it commutes with left and right multiplication by the elements from $\mathcal{A}^c_{0\,q}$ in the following sense:

$$\langle fP, Q\rangle = f\langle P, Q\rangle \qquad \langle P, Qf\rangle = \langle P, Q\rangle f \qquad \forall\, f \in \mathcal{A}^c_{0\,q} \qquad P \in T(H_q)_l \qquad Q \in T(H_q)_r \tag{4.5}$$

(in particular, $P,\ Q \in V'$) and if it is compatible with the action of $U_q(sl(2))$. The latter property means, as usual, that

$$Z\langle , \rangle = \langle , \rangle \Delta(Z) \qquad \forall Z \in U_q(sl(2))$$

(this relation is treated as operator one in $T(H_q)_l \otimes_k T(H_q)_r$). A metric is called symmetric if

$$\langle , \rangle (V' \otimes V')_1 = 0. \tag{4.6}$$

Proposition 7. *There exists the unique (up to a factor) symmetric quantum metric on the quantum hyperboloid.*

A proof of this fact is given in [A]. We do not reproduce it here. Let us indicate only the crucial idea of the proof. First, it is necessary to describe all pairings

$$\langle , \rangle : V' \otimes V' \to \mathcal{A}^c_{0\,q}$$

compatible with the $U_q(sl(2))$ action. In order to do it we should decompose $V' \otimes V'$ into a sum of the irreducible $U_q(sl(2))$-modules. This gives rise to the following two-parameter family of $U_q(sl(2))$-covariant pairing

$$\langle , \rangle (V' \otimes V')_2 = aV_2 \qquad \langle , \rangle (V' \otimes V')_0 = b$$

completed by relations (4.6) (as usual, the relations are given in a symbolic way). On the second step we should impose the condition

$$\langle,\rangle^{23}(V \otimes V')_0 \otimes V' = 0$$

which results in a relation between the parameters a and b. It remains to verify that this relation is compatible with the following one

$$\langle,\rangle^{12} V' \otimes (V' \otimes V)_0 = 0$$

and then to extend the metric to the whole $T(H_q)_l \otimes_k T(H_q)_r$ by using the relations (4.5).

Let us emphasize that although we call the above pairing metric ('pseudo' means only that its classical analogue is not positive definite) it is well defined on the product of a left and a right $\mathcal{A}^c_{0\,q}$-modules. If we want now to define a similar pairing between two left (or right) $\mathcal{A}^c_{0\,q}$-modules we should proceed in the following way. Let us identify the left tangent space $T(H_q)_l$ and the right one $T(H_q)_r$, i.e. define a map

$$\alpha : T(H_q)_l \to T(H_q)_r$$

being a $U_q(sl(2))$-isomorphism.

Then on setting by definition

$$\langle X, Y\rangle = \langle X, \alpha(Y)\rangle \qquad X, Y \in T(H_q)_l$$

we get a pairing between two left $U_q(sl(2))$-modules. At first glance the map α can be defined by means of the YB operator arising from the QG $U_q(sl(2))$. However, this operator which establishes a bijectivity between free $\mathcal{A}^c_{0\,q}$-modules $\mathcal{A}^c_{0\,q} \otimes V'$ and $V' \otimes \mathcal{A}^c_{0\,q}$ is not any bijectivity on their factors $T(H_q)_l$ and $T(H_q)_r$ since it does not take the denominator corresponding to $T(H_q)_l$ to that corresponding to $T(H_q)_r$. We suggest another way to define such a $U_q(sl(2))$-isomorphism α.

Let us represent the both objects as sums of $U_q(sl(2))$-modules in the spirit of proposition 1. Then the $U_q(sl(2))$-morphisms

$$\alpha : (V \otimes V')_1 \to (V' \otimes V)_1 \qquad (V \otimes V')_2 \to (V' \otimes V)_2$$
$$(V_2 \otimes V')_2 \to (V' \otimes V_2)_2 \ldots$$

are defined uniquely up to a factor on each couple of components (for the generating space V' we put $\alpha = \mathrm{id}$).

However, we can reduce this freedom by identifying the elements from

$$(V \otimes V')_2 \quad \text{and} \quad (V' \otimes V)_2 \qquad (V_i \otimes V')_{i+1} \quad \text{and} \quad (V' \otimes V_i)_{i+1} \qquad i = 2, 3, \ldots$$

which coincide if we replace V' by V. As for the components

$$(V \otimes V')_1 \quad \text{and} \quad (V' \otimes V)_1 \qquad (V_i \otimes V')_i \quad \text{and} \quad (V' \otimes V_i)_i \qquad i = 2, 3, \ldots$$

their elements are identified if this operation leads to opposite images. It is not difficult to see that in the classical case this identification and that defined by the flip coincide (it is the motivation of our method).

Remark 3. *Let us remark that for algebras looking like that $\mathcal{A}^c_{0\,q}$ but connected to an involutory YB operator an identification of their left and right modules can be realized by means of this operator. Non-involutivity of the YB operator arising from the QG $U_q(sl(2))$ which leads to the above defect prevents us also from a reasonable definition of a tensor product $M_1 \otimes_{\mathcal{A}^c_{0_q}} M_2$ of two (say) left $\mathcal{A}^c_{0\,q}$-modules. The problem is that there do not exist any reasonable way to transpose the factor $f \in \mathcal{A}^c_{0\,q}$ in the product*

$$m_1 \otimes f\, m_2 \qquad m_1 \in M_1 \qquad m_2 \in M_2$$

on the left side so that the tensor product $\otimes_{\mathcal{A}^c_{0_q}}$ is still associative and the module $M_1 \otimes_{\mathcal{A}^c_{0_q}} M_2$ is a flat deformation of its classical counterpart assuming M_1 and M_2 to be flat deformations of their classical counterparts. For an involutory YB operator this problem does not appear.

Let us now discuss the problem of defining a (torsion-free) $U_q(sl(2))$-covariant connection on the tangent space $T(H_q)$. Such a partially defined connection was introduced in [A]. 'Partially defined' means here that the operators of covariant derivatives are defined only on a subspace of $T(H_q)$, namely on V'. More precisely, there exists a $U_q(sl(2))$-morphism

$$\nabla : T(H_q)_l \otimes V' \to T(H_q)_l$$
$$X \otimes Y \mapsto \nabla_X Y$$

such that

$$\nabla_{fX} Y = f \nabla_X Y \qquad X \in T(H_q) \qquad Y \in V' \quad f \in \mathcal{A}^c_{0\,q}$$

and

$$a^{ij} \nabla_{X_i} X_j = a^{ij} [X_i, X_j]_q \qquad X_i, X_j \in V' \tag{4.7}$$

where $a^{ij} X_i \otimes X_j \in V_1$ and $[\ ,\ ]_q$ is the above-mentioned q-deformed Lie bracket.

We would be able to extend this (partially defined) connection to the whole $T(H_q) \otimes T(H_q)$ if we could extend the bracket $[\ ,\ ]_q$ to the whole $T(H_q)$ and to understand what is the enveloping algebra of this extended q-deformed Lie algebra (we need this in order to write suitable expressions in the lhs of (4.7)). Unfortunately, we do not know any way to do this.

Let us remark that all naive extensions of the bracket $[\ ,\ ]_q$ are not compatible with the equation (4.2).

There exist a number of papers introducing the notions of quantum metric and connection in another way (cf [HM] and the references therein). Our approach is motivated by our desire to control the flatness of deformation of classical objects (see also the next section).

Completing this section, we want to mention a very important property of the $\mathcal{A}^c_{0\,q}$-module $T(H_q)$ (if $c \neq 0$): it is projective in the category of $U_q(sl(2))$-modules. This means that it is a direct component in a free $\mathcal{A}^c_{0\,q}$-module and there exists a projector of the latter one onto the module in question being a $U_q(sl(2))$-morphism (cf [A]).

Some other projective modules over quantum sphere have been considered in [HM]. We plan to devote a subsequent paper to quantum projective modules in a more general context.

5. On quantum gauge theory

There exist two approaches to q-deformed gauge theory. One of them deals with the usual manifolds (varieties) and deforms only a structure of fibres. The second approach deals with quantum varieties looking like the quantum hyperboloid above.

First, let us evoke the paper [S] as the most advanced contribution to the first kind of approach. The gauge potential A_μ introduced in this paper is a vector field

$$A_\mu(x) = A^i_\mu(x) X_i$$

with values in the quantum Lie algebra $\boldsymbol{g}_q$. Here $\boldsymbol{g}_q$ is the q-deformed Lie algebra $\boldsymbol{g} = sl(n)$ (or $su(n)$) as defined in [LS] (note that the case $n = 2$ was previously considered in [DG1]). Thus, the factors X_i are elements of this quantum Lie algebra and those $A^i_\mu(x)$ are usual functions depending on a 'space-time point' x (or more generally, on a point of a usual variety).

In virtue of [LS] the quantum Lie algebra $\boldsymbol{g}_q$ is realized as a subspace in $U_q(\boldsymbol{g})$ so that it is stable w.r.t. the adjoint action of the QG $U_q(\boldsymbol{g})$ on itself and

$$\Delta(X_i) = X_i \otimes C + u^j_i \otimes X_j$$

where Δ is the coproduct in $U_q(\boldsymbol{g})$, C is a central (Casimir) element of $U_q(\boldsymbol{g})$ and u^j_i are some elements of $U_q(\boldsymbol{g})$.

The crucial point of any gauge theory is a transformation law of A_μ under an action of a gauge group or a gauge Hopf algebra. In [S] it is supposed to be

$$A_\mu \mapsto A'_\mu(x) = h(x)_{(1)} A_\mu s(h(x)_{(2)}) - \alpha^{-1} s(C)^{-1} \partial_\mu (h(x)_{(1)}) s(h(x)_{(2)}) \tag{5.1}$$

where α is a coupling constant, s is the antipode, $h(x)$ is a function of x with values in the QG $U_q(\boldsymbol{g})$ and

$$h(x)_{(1)} \otimes h(x)_{(2)} = \Delta h(x).$$

However, a problem arises to 'distribute the x-dependence of the coproduct $\Delta(h(x))$ between two factors' in the second term of (5.1) so that it becomes an element of the space $\boldsymbol{g}_q$ for a fixed x (let us emphasize that it is not a trivial task). Such a distribution has been found in [S].

Nevertheless, it was indicated in [S] that if we consider a (say) bosonic field $\psi(x)$ then its defining relations cannot be introduced in a way compatible with quantum gauge transformations. We will try to explain this as follows. Let us consider the 'quantum covariant derivative' of the field ψ

$$D_\mu \psi = \partial_\mu \psi + \beta \rho(A_\mu) \psi$$

where ρ is the representation of the QG $U_q(\boldsymbol{g})$ corresponding to ψ and $\beta = \alpha\rho(s(C))$ is a constant. In the rhs of this formula the operator ∂_μ commutes with the $U_q(\boldsymbol{g})$ action but the operator $\rho(A_\mu)$ does not. This implies that such a covariant derivative cannot preserve the relations valid for ψ. (We would have a similar effect in a supertheory if we allowed the summand A_μ to be an odd operator.)

Let us discuss now the approach of the second kind, i.e. we suppose that the base variety is quantum as well. An axiomatic way to suggest such an approach was considered in numerous papers. We do not give here an exhausting list of these papers and only refer the reader to the papers [S] and [BM] where such a list is given.

We will point out the common features of all of them. First, a quantum variety in question is given in a way which does not allow us to control the flatness of deformation (as a rule this problem is not even evoked). Another crucial defect of this approach is that a connection is introduced habitually via a Leibniz rule similarly to the classical case but as we have seen this implied the non-flatness of the deformation. Another reason of the non-flatness of the deformation is that in the formulae analogical to (5.1) the second summand does not belong usually to the fibre.

This explains our scepticism about a possibility of constructing a quantum gauge theory related to the QG $U_q(\boldsymbol{g})$ which would be a flat deformation of its classical counterpart.

Anyway, it would be desirable to precede any attempt to construct such a theory by a quasiclassical study in the spirit of [Ar] confirming or refuting the possibility of doing it.

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